Reducing Fuel Consumption of Subsonic Aircraft by Optimal Cyclic Cruise

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Fuel consumption in range and endurance flight is considered as an optimal cyclic control problem. In regard to range cruise, the incompressible and compressible flight regimes are treated separately because each of them shows specific effects for optimal cyclic flight. The improvements achievable in the incompressible flight regime depend on the admissible altitude range. For the compressible flight regime, it is shown that drag rise effects represent a key factor limiting the improvements possible by optimal cyclic cruise. Furthermore, results are presented for endurance flight, which is more improved by optimal cyclic control than is range cruise.

Nomenclature

C_D = drag coefficient
C_L = lift coefficient
D = drag
E = energy
G = function denoting state variable constraint
g = acceleration due to gravity
H = Hamiltonian
h = altitude
J = performance criterion
L = lift
M = Mach number
m = mass
S = reference area
T = thrust
t = time
V = speed
x = horizontal coordinate
γ = flight path angle
δ = throttle setting
λ, μ, ν = Lagrange multipliers
ρ = atmospheric density
σ = fuel consumption gradient with regard to thrust

Introduction

THERE are many efforts to reduce the fuel consumption in order to increase the flight efficiency and performance of aircraft. Some of these are concerned with trajectory optimization. Recent results show that the well-known steady-state cruise is not generally optimal but that improvements may be achieved by a nonsteady type of cruise.[1-16] This type of cruise consists of a flight path where the control and state variables behave in a periodic manner. As a result, the trajectory of the aircraft is no longer rectilinear but shows a periodic behavior consisting of repeated cycles.

It is the purpose of this paper to develop further the understanding of optimal cyclic cruise and its possible superiority to steady-state cruise. In particular, it will be shown for the range cruise problem that optimal cyclic control in the incompressible flight regime may be regarded as a problem different from the compressible flight regime. For endurance maximization, the incompressible flight regime appears to be of predominant importance.

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Problem Formulation

The range cruise problem is to find periodic flight paths where the fuel consumed per range traveled is smaller than for the best steady-state cruise. This is equivalent to a periodic control problem consisting of minimizing the following performance criterion:

$$J = -\frac{x_{cyc}}{m_f(x_{cyc})} \tag{1}$$

The expression shown represents the ratio of a horizontal cycle length x_{cyc} to the fuel consumed in a cycle $m_f(x_{cyc})$.

For endurance flight, the performance criterion may be written as

$$\bar{J} = -\frac{t_{cyc}}{m_f(t_{cyc})} \tag{2}$$

Both criteria are subject to the equations of motion. Since the expressions developed herein are concerned with optimizing range cruise, the equations are written in a way suitable for this problem by using the horizontal coordinate x as the independent variable. For optimizing endurance flight, the time may be retained as the independent variable and similar expressions not given here may be developed.

The equations of motion and fuel consumption may be written as

$$\frac{dV}{dx} = \frac{T - D - mg\sin\gamma}{mV\cos\gamma} \qquad \frac{d\gamma}{dx} = \frac{L - mg\cos\gamma}{mV^2\cos\gamma}$$
$$\frac{dh}{dx} = \tan\gamma \qquad \frac{dm_f}{dx} = \frac{\dot{m}_{f_0} + \sigma T}{V\cos\gamma} \tag{3}$$

The mass of the airplane can be considered constant for one cycle, since the fuel consumed is small as compared with the total of the mass, i.e.,

$$m_f(x_{cyc}) - m_f(0) \ll m \tag{4}$$

The periodicity of the flight path implies the following boundary conditions:

$$V(x_{cyc}) = V(0), \quad \gamma(x_{cyc}) = \gamma(0), \quad h(x_{cyc}) = h(0) \tag{5}$$

The initial condition for the fuel mass may be written as

$$m_f(0) = 0 \tag{6}$$

The models for thrust, lift, and drag are

$$T = T_{\min}(h) + \delta[T_{\max}(h) - T_{\min}(h)]$$

$$L = C_L(\rho/2)V^2S$$

$$D = C_D(\rho/2)V^2S \tag{7}$$

The drag polar used for the incompressible case is of symmetric type according to

$$C_D = C_{D_0} + kC_L^2 \tag{8}$$

The maximum lift-to-drag ratio for the aircraft considered is given by $(L/D)_{\max} = 18.3$. The thrust model for the incompressible case is

$$T_{\max} = (T_{\max})_{h=0}(\rho/\rho_0)^{0.7}$$

$$T_{\min} = (T_{\min})_{h=0}(\rho/\rho_0)^{0.7} \tag{9}$$

where the subscript 0 denotes the reference values at $h = 0$. The specific fuel consumption is considered constant for the speed altitude range of interest (σ = const, $\dot{m}_{f_0} = 0$).

For the compressible case, the drag polar characteristics fully account for the dependence of the drag coefficient on the lift coefficient and Mach number, $C_D = C_D(C_L, M)$. This is illustrated in Fig. 1, which gives an indication of the kind of accuracy achieved with the modeling. In a similar way, the powerplant characteristics are modeled. This is illustrated in Fig. 2, which shows the thrust and fuel consumption characteristics for the aircraft used in the compressible case. It may be added that the dependence of fuel consumption characteristics on the Mach number is also included in the modeling, yielding a similar accuracy as shown in Fig. 2 for $M = 0.8$. The atmospheric model used for the air density, speed of sound, and thrust dependence on altitude corresponds to the ICAO Standard Atmosphere.[17]

The control variables are the lift coefficient C_L and the throttle setting δ, which are subject to the following inequality constraints:

$$C_{L_{\min}} \le C_L \le C_{L_{\max}}$$

$$0 \le \delta \le 1 \tag{10}$$

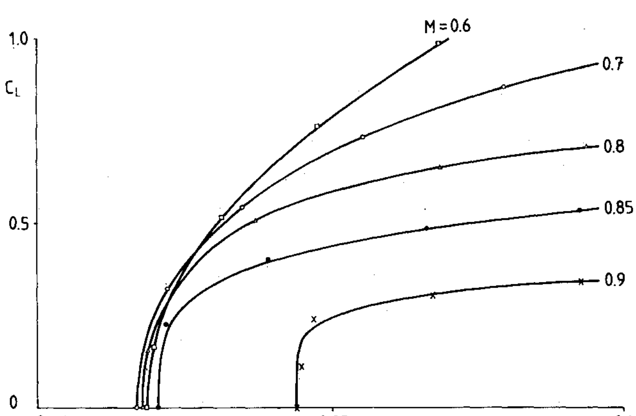

Fig. 1 Modeling of drag polar for the aircraft used in the compressible case (given aircraft data indicated by individual symbols).

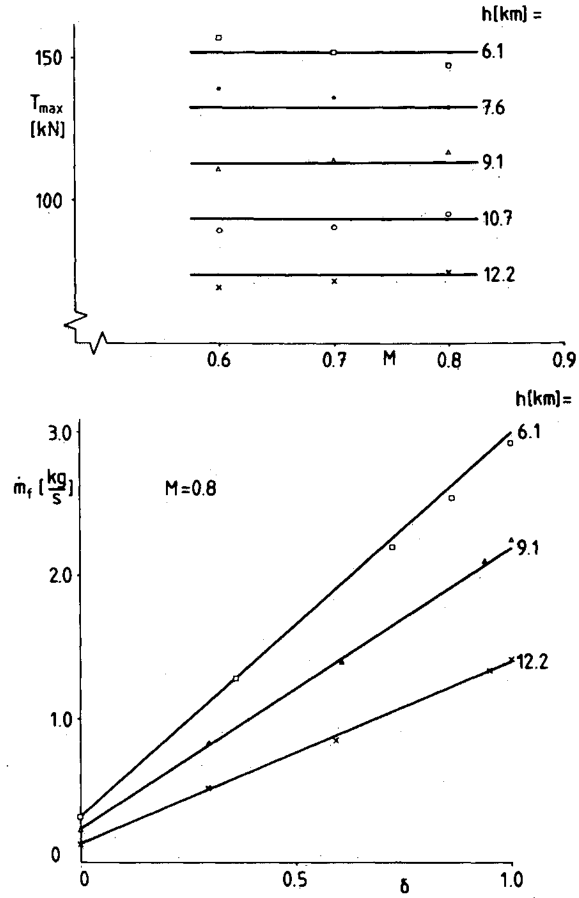

Fig. 2 Modeling of thrust and fuel consumption characteristics for the aircraft used in the compressible case (given aircraft data indicated by individual symbols), $T_{\min} = 0$.

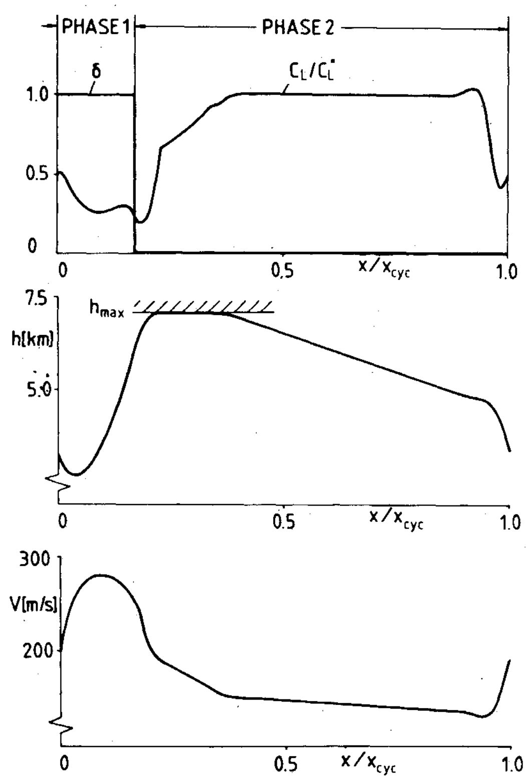

Fig. 3 Basic characteristics of optimal cyclic cruise (turbojet-type powerplant, $(T_{\max})_{h=0} = 0.5$ mg, $x_{cyc} = 81.2$ km, C_L^*: lift coefficient corresponding to the maximum lift-to-drag ratio); 16% reduction of fuel consumption.

The periodic control problem can now be stated as the finding of the control histories C_L and δ, the initial states $[V(0), \gamma(0), h(0)]$, and the periodic cycle length x_{cyc} that minimize the performance criterion $J = -x_{cyc}/m_f(x_{cyc})$ subjected to the dynamic system described by Eq. (3), the boundary conditions given by Eq. (5), and the inequality constraints of Eq. (10) for the control variables.

Optimality Conditions

Necessary conditions for optimality can be determined by applying the minimum principle. For this purpose, the Hamiltonian is defined as

$$H = \lambda_V \frac{T - D - mg\sin\gamma}{mV\cos\gamma} + \lambda_\gamma \frac{L - mg\cos\gamma}{mV^2\cos\gamma} + \lambda_h \tan\gamma + \lambda_f \frac{\dot{m}_{f0} + \sigma T}{V\cos\gamma} \tag{11}$$

where the Lagrange multipliers $\lambda^T = (\lambda_V, \lambda_\gamma, \lambda_h, \lambda_f)$ have been adjoined to the dynamic system of Eq. (3). The Lagrange multipliers are determined by‡

$$\frac{d\lambda_V}{dx} = \lambda_V \frac{T - D - mg\sin\gamma + VD_V}{mV^2\cos\gamma} + \lambda_\gamma \frac{2(L - mg\cos\gamma) - VL_V}{mV^3\cos\gamma} + \lambda_f \frac{\dot{m}_{f0} - V\partial\dot{m}_{f0}/\partial V + T(\sigma - V\sigma_V)}{V^2\cos\gamma}$$

$$\frac{d\lambda_\gamma}{dx} = \lambda_V \frac{(D-T)\sin\gamma - mg}{mV\cos^2\gamma} - \lambda_\gamma \frac{L\sin\gamma}{mV^2\cos^2\gamma} - \frac{\lambda_h}{\cos^2\gamma} - \lambda_f \frac{(\dot{m}_{f0} + \sigma T)\sin\gamma}{V\cos^2\gamma}$$

$$\frac{d\lambda_h}{dx} = \lambda_V \frac{D_h - T_h}{mV\cos\gamma} - \lambda_\gamma \frac{L_h}{mV^2\cos\gamma} - \lambda_f \frac{\partial\dot{m}_{f0}/\partial h + \sigma T_h + \sigma_h T}{V\cos\gamma}$$

$$\frac{d\lambda_f}{dx} = 0 \tag{12}$$

with the following boundary conditions:

$$\lambda_V(x_{cyc}) = \lambda_V(0), \qquad \lambda_\gamma(x_{cyc}) = \lambda_\gamma(0),$$
$$\lambda_h(x_{cyc}) = \lambda_h(0), \qquad \lambda_f(x_{cyc}) = x_{cyc}/m_f^2(x_{cyc}) \tag{13}$$

The optimal controls C_L and δ are such that H is minimized. For this reason, C_L is determined either by

$$\frac{\partial H}{\partial C_L} = 0 \tag{14}$$

or by the constraining bounds of Eq. (10). In regard to the throttle setting δ, H is considered linear in δ. Thus, δ shows a bang-bang type of behavior:

$$\delta = 0 \text{ for } H_\delta > 0$$
$$\delta = 1 \text{ for } H_\delta < 0 \tag{15}$$

‡Partial derivatives of D, H, L, T, and σ are denoted by subscripts, e.g., $D_V = \partial D/\partial V$.

It is assumed that there are no singular arcs where δ takes on intermediate values. The occurrence of a singular arc would require $H_\delta = 0$ for a finite interval of range. Such a behavior was not observed in the numerical investigation.

The system described by Eq. (3) is autonomous, so that the Hamiltonian H is constant. Since, furthermore, the cycle length x_{cyc} is considered free, H is given by

$$H = 1/m_f(x_{cyc}) \tag{16}$$

Optimal Flight Paths with Altitude Constraints

There are cases of cyclic range cruise optimization where it is necessary to introduce an upper bound for the altitude range admissible. Then cyclic cruise must be considered as a periodic optimization problem with a state variable constraint $h \le h_{max}$. As a consequence, there are additional conditions presented herein according to Ref. 18.

Basically, two possibilities exist in regard to the constraint under consideration. One is concerned with the flight path touching the constrained altitude boundary at only one point. The other possibility is characterized by the fact that the optimal flight path stays on the altitude boundary for a finite interval. Both possibilities have been observed in the numerical investigation, so the additional conditions for each of them are presented herein.

The altitude constraint can be formulated as

$$G(h) = h - h_{max} \le 0 \tag{17}$$

Since

$$G^{(1)}(\gamma) = \tan\gamma \tag{18a}$$

$$G^{(2)}(V, \gamma, h; C_L) = \frac{L - mg\cos\gamma}{mV^2\cos^3\gamma} \tag{18b}$$

the state variable constraint is of second order.

In regard to the first possibility previously described, Eqs. (17) and (18a) yield the following conditions:

$$h(x_1) = h_{max}$$
$$\gamma(x_1) = 0 \tag{19}$$

with x_1 denoting the point where the flight path touches the constraint altitude. At this point, λ_h shows a discontinuous change. Denoting by x_1^- a point just before the point under consideration and by x_1^+ a point immediately after, the following relation holds:

$$\lambda_h(x_1^+) = \lambda_h(x_1^-) + \nu_0 \tag{20}$$

where $\nu_0 \le 0$.

The other possibility described shows a constrained arc where the optimal flight path stays on the altitude boundary. On the constrained arc, the Hamiltonian is changed to

$$H \to H + \mu(x) G^{(2)}(V, \gamma, h; C_L) \tag{21}$$

Consequently, the following relations for the Lagrange multipliers exist:

$$\frac{d\lambda_V}{dx} = -\frac{\partial H}{\partial V} - \mu(x)\frac{\partial}{\partial V}G^{(2)}$$

$$\frac{d\lambda_\gamma}{dx} = -\frac{\partial H}{\partial \gamma} - \mu(x)\frac{\partial}{\partial \gamma}G^{(2)}$$

$$\frac{d\lambda_h}{dx} = -\frac{\partial H}{\partial h} - \mu(x)\frac{\partial}{\partial h}G^{(2)} \tag{22}$$

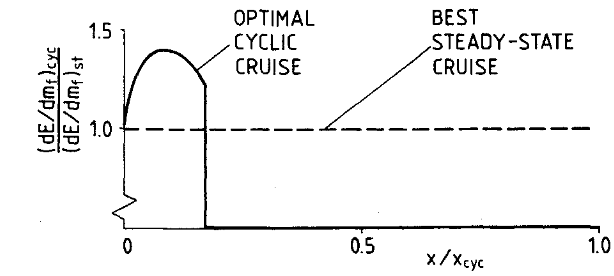

Fig. 4 Energy added per fuel consumed.

The equation for λ_f remains unchanged since $G^{(2)}$ is independent of m_f.

The relation for $\mu(x)$ on the constrained arc is (from $H_{C_L}=0$ and $\gamma=0$)

$$\mu(x)=\lambda_V V\frac{\partial C_D}{\partial C_L}-\lambda_\gamma \tag{23}$$

On the unconstrained arc, $\mu(x)=0$.

In regard to the optimal controls on the constrained arc, Eq. (18b) yields the following relation for C_L (with $\gamma=0$):

$$C_L=\frac{2mg}{\rho V^2 S} \tag{24}$$

This represents the lift equation $L=mg$ for accelerated/decelerated horizontal flight. In regard to the entry point x_1 of the constrained arc, Eqs. (17), (18a), and (18b) yield the following conditions:

$$h(x_1)=h_{\max}$$

$$\gamma(x_1)=0$$

$$C_L-\frac{2mg}{V^2 S}=0 \tag{25}$$

with C_L corresponding to Eq. (14).

Some of the Lagrange multipliers show discontinuous changes at the entry point. The following conditions hold:

$$\lambda_h(x_1^+)=\lambda_h(x_1^-)+\nu_0$$

$$\lambda_\gamma(x_1^+)=\lambda_\gamma(x_1^-)+\nu_1 \tag{26}$$

where ν_0 and ν_1 represent two additional unknowns. The Lagrange multipliers λ_V and λ_f are continuous at the entry point.

In the numerical investigation, an optimization program based on the method of multiple shooting was applied. In these references, the numerical procedure is described in detail.

Characteristics of Optimal Cyclic Flight

In Fig. 3, an optimal cycle for range maximization per fuel consumed is shown in order to illustrate the basic characteristics of periodic cruise flight. As indicated in this figure, an optimal cycle may be decomposed into two phases that can be characterized by thrust behavior. In the first phase, thrust is at its maximum which, due to $T_{\max}>D$, results in an increase of the energy state of the aircraft. The second phase shows thrust at its minimum, where the energy state increase of the preceding phase is used to gain as much range as possible. Corresponding to the thrust behavior, the speed level in phase 1 is high compared with that of phase 2, and the altitude indicating the potential energy level shows an increase in phase 1 and a decrease in phase 2.

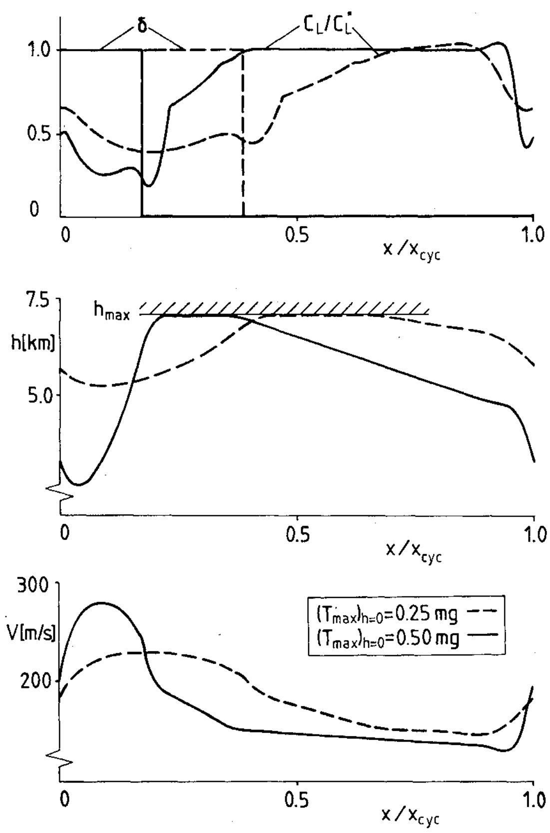

Fig. 5 Effect of maximum thrust available on optimal cyclic trajectories (C_L^*: lift coefficient corresponding to the maximum lift-to-drag ratio).

The behavior just described can give a physical insight into the reasons why cyclic cruise can provide a reduction in fuel consumption. This is illustrated in Fig. 4, which shows the energy added per fuel consumed for optimal cyclic cruise and for the best steady-state flight in the altitude range considered. Due to the high speed level in phase 1, it is possible to reach values of $(dE/dm_f)_{cyc}$ that are considerably better than the best values of steady-state cruise $(dE/dm_f)_{st}$.

In the following sections, improvements due to cyclic cruise are shown and decisive aircraft factors are identified and evaluated. For this purpose, it is suitable to treat the incompressible and compressible flight regimes separately, since the physical effects underlying optimal cyclic flight in both regimes are quite different.

Optimal Cyclic Cruise in the Incompressible Flight Regime

The maximum thrust level available represents a key factor as regards cyclic cruise and its possible improvements. This is already indicated by the thrust behavior outlined in the description of the basic characteristics of optimal cyclic cruise, as illustrated in Figs. 3 and 4. From the thrust behavior described, it follows that the improved fuel utilization for increasing the energy state results because excess thrust is available. An example of the effect of the maximum thrust available on the trajectory is shown in Fig. 5. Characteristically, the higher thrust level results in larger amplitudes of the changes of speed and altitude, thus showing a more pronounced oscillatory behavior.

An evaluation of the effect of maximum thrust on the improvements achievable is shown in Fig. 6. From the results presented, it follows that the gains can be significantly increased when more thrust is available. This corresponds to similar conclusions given in Ref. 10. Figure 6 also includes the

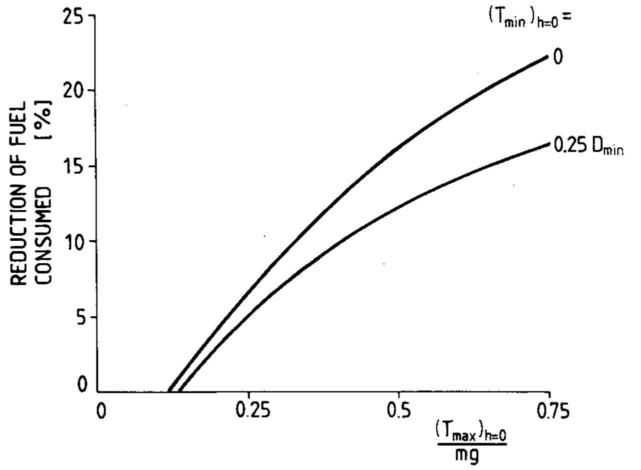

Fig. 6 Effect of maximum thrust weight ratio on the reduction of fuel consumed (turbojet-type power plant).

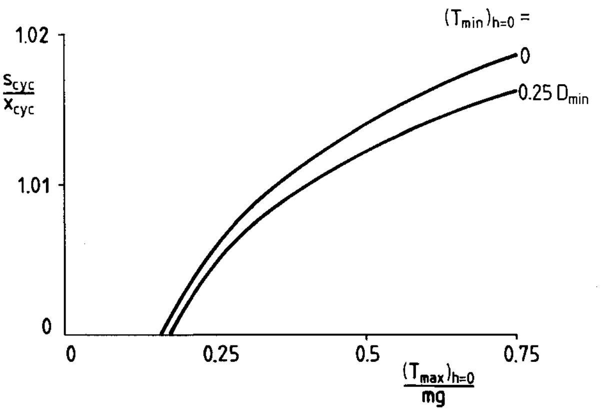

Fig. 7 Increase of flight path length (s_{cyc}) in comparison to the horizontal distance traveled (x_{cyc}).

effect of a nonzero minimum thrust phase. The minimum thrust is related to the minimum drag of a steady-state horizontal flight $D_{min} = mg(C_D/C_L)_{min}$. Figure 6 shows that the minimum thrust level also has a significant effect on the gains achievable, which are reduced when the minimum thrust is increased.

It may be of interest to note how much the length of the optimal cyclic trajectory is increased when compared with the horizontal distance traveled, which also represents the length of the steady-state cruise trajectory. This is illustrated in Fig. 7, where s_{cyc} is the actual flight path length of cyclic cruise and x_{cyc} the corresponding horizontal distance traveled. Figure 7 shows that there is some length increase that becomes large when more thrust is available. This results because a higher thrust level yields a more pronounced oscillatory flight profile behavior.

In the results presented so far, an altitude constraint of $h \leq h_{max}$ is imposed. This indicates a tendency for the optimal altitude range to be as high as possible. In regard to steady-state cruise, an increase in the admissible altitude range usually also yields an improvement (with the best steady-state solution on the h_{max} boundary and the compressible flight regime excluded). It is therefore of interest to know how the optimal cyclic cruise compares to the best steady-state cruise when increasing the admissible altitude range. This is illustrated in Figs. 8 and 9. Figure 8 shows flight profiles for various values of h_{max}. Figure 9 presents an evaluation of the effect of the altitude constraint. From this it follows that the improvements achievable with cyclic cruise are reduced when the altitude range is increased.

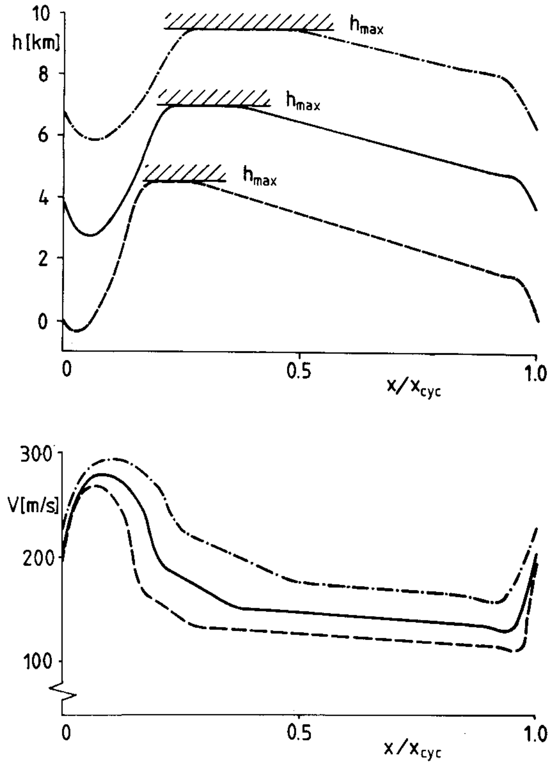

Fig. 8 Effect of an altitude constraint on optimal cyclic trajectories.

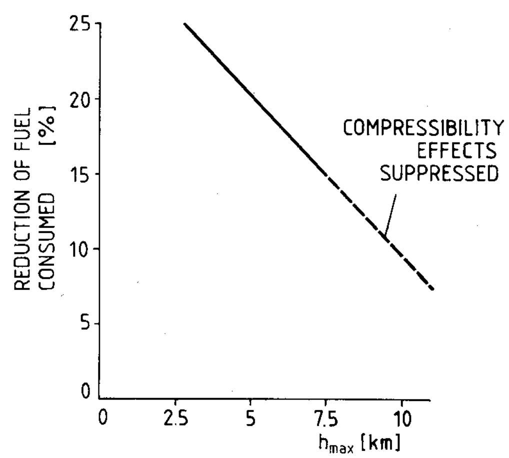

Fig. 9 Effect of an altitude constraint on fuel consumption reduction due to optimal cyclic flight.

The tendency of the curve of Fig. 9 suggests that the improvement due to cyclic cruise may disappear if the admissible altitude is high enough. From a practical standpoint, however, this effect may be of minor importance when compared with the consequences resulting from the fact that the altitude range increase leads to higher speeds that may eventually approach the compressible flight regime (as indicated in Fig. 9). In such a case, an altitude constraint changes its meaning for the optimal cycle and may even not become active, despite sufficient excess thrust, since the drag rise due to compressibility may predominate.

Optimal Cyclic Cruise in the Compressible Flight Regime

For steady-state cruise, it is well known that the drag rise due to compressibility in the high subsonic Mach number range limits the maximum Mach number economically usable.

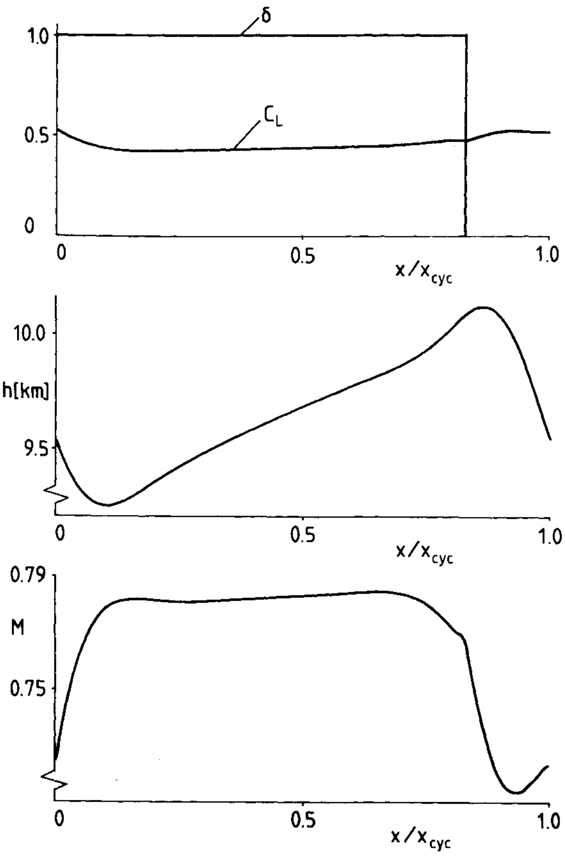

Fig. 10 Optimal cyclic cruise in the compressible flight regime ($x_{cyc} = 76.4$ km); 0.2% reduction of fuel consumed compared to the best steady-state cruise.

It is therefore an effect of primary importance for all flight vehicles at subsonic speed. It will be shown that the drag rise due to compressibility also represents a key factor for cyclic cruise, where it also acts as a kind of barrier.

Figure 10 presents an example that shows the optimal cycle of the aircraft illustrated in Figs. 1 and 2 with the altitude constraint removed and no other constraint imposed. There are again maximum and minimum thrust phases that can be considered as basic elements of cyclic flight, as described earlier. However, the oscillatory behavior in terms of the maximum changes of the Mach number and lift coefficient is reduced. In particular, the maximum Mach number attained appears to be limited, despite the fact that no Mach number constraint is imposed. The reason for this type of effective Mach number limitation is due to aerodynamic characteristics, because compressibility effects yield a substantial drag rise in the Mach number range of interest. As a consequence, the results for cyclic cruise in the compressible Mach number range show only small improvements when compared with results for the incompressible flight regime, as described earlier.

Optimal Cyclic Endurance Flight

For the range cruise problem, the improvements possible by cyclic control appear to be reduced when no altitude constraint is imposed or when the compressible flight regime is approached. By contrast, endurance flight may be significantly improved by cyclic control. An example is presented in Fig. 11, which shows the histories of state and control variables. There are similarities as regards the maximum and minimum thrust phases and the corresponding climbing and sinking flight conditions. However, there are also significant differences. The speed attained remains within the incompressible flight regime. This means, for the aircraft models considered, that a separate treatment of the compressible flight

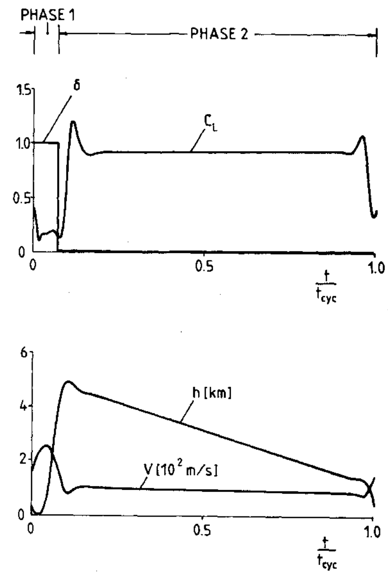

Fig. 11 Optimal cyclic endurance flight $[(T_{max})_{h=0} = 0.5$ mg, $t_{cyc} = 11.87$ min]; 63.5% endurance increase compared to the best steady-state flight.

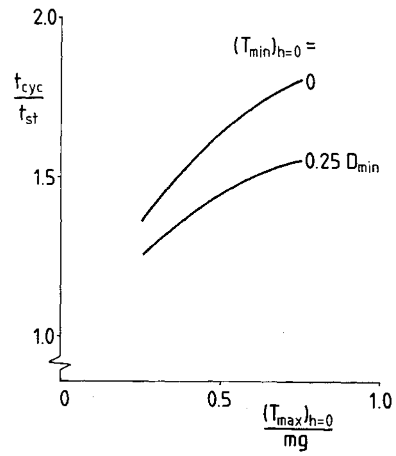

Fig. 12 Effect of maximum thrust weight ratio on endurance increase due to optimal cyclic control.

regime is not necessary. Another difference concerns the altitude boundary. In the numerical investigation, it was observed that the greatest endurance per fuel consumed was reached for altitudes as low as possible. Therefore, it was necessary to introduce a lower bound for the admissible altitude range.

The maximum thrust level is again of great influence. This is illustrated in Fig. 12, which shows the endurance of optimal cyclic flight (t_{cyc}) as compared to the best steady-state flight (t_{st}) for the same amount o fuel. From the results presented, it follows that the improvements of optimal cyclic control are significant, even for comparatively low thrust levels.

Conclusions

The minimization of fuel consumption for a given range is considered as an optimal cyclic control problem, partially with a state variable constraint given by an upper altitude bound. The incompressible and compressible flight regimes are treated separately because each of them shows specific effects.

In regard to the basic characteristics of cyclic cruise, an optimal cycle may be decomposed into two phases of which one is a maximum-thrust increasing-energy condition and the other a minimum-thrust decreasing-energy condition. The improvements of optimal cyclic cruise are due to better energy management.

In the incompressible flight regime, the improvements depend on the admissible altitude range. They are reduced when the thrust level is reduced or the maximum altitude admissible is increased.

In regard to the compressible flight regime, it is shown that cyclic cruise can provide an improvement which is, however, comparatively small. The drag rise due to compressibility is identified as a key factor limiting the possibilities of cyclic cruise at high subsonic Mach numbers.

Endurance flight may be more improved by cyclic control than range cruise. It is shown that the maximum thrust available is, again, of great influence.

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